

# NON SEMI-SIMPLE $\mathfrak{sl}(2)$ QUANTUM INVARIANTS, SPIN CASE

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ABSTRACT. Invariants of 3-manifolds from a non semi-simple category of modules over a version of quantum  $\mathfrak{sl}(2)$  were obtained by the last three authors in [8]. In their construction the quantum parameter  $q$  is a root of unity of order  $2r$  where  $r > 1$  is odd or congruent to 2 modulo 4. In this paper we consider the remaining cases where  $r$  is congruent to zero modulo 4 and produce invariants of 3-manifolds with colored links, equipped with generalized spin structure. For a given 3-manifold  $M$ , the relevant generalized spin structures are (non canonically) parametrized by  $H^1(M; \mathbb{C}/2\mathbb{Z})$ .

## INTRODUCTION

New quantum invariants of 3-manifolds equipped with 1-dimensional cohomology class over  $\mathbb{C}/2\mathbb{Z}$  or equivalently  $\mathbb{C}^*$  flat connection have been constructed in [8] from a variant of quantum  $\mathfrak{sl}(2)$ . This family of invariants is indexed by integers  $r \geq 2$ ,  $r \not\equiv 0 \pmod{4}$ , which give the order of the quantum parameter. The relevant representation category is non semi-simple, so that the usual modular category framework does not apply and is replaced by more general *relative  $G$ -modular category*. The required non degeneracy condition is not satisfied in cases  $r \equiv 0 \pmod{4}$ . In the present paper we show that the procedure can be adapted to the remaining cases and leads to invariants of 3-manifolds with colored links, equipped with some generalized spin structure. These spin structures can be defined as certain cohomology classes on the tangent framed bundle and can be interpreted as  $C^*$  flat connections on this framed bundle.

The non semi-simple  $\mathfrak{sl}(2)$  invariants from [8] have been extended to TQFTs in [6]. For  $r = 2$ , they give a TQFT for a canonical normalization of Reidemeister torsion; in particular they recover classification of lens spaces. In general they give new representations of Mapping Class Groups with opened faithfulness question. They contain the Kashaev invariants and give an extended formulation of the volume conjecture. Similar TQFTs in the spin case would be interesting to construct and study.

Similarly to the cohomological case studied in [8], we define in this paper a secondary invariant for empty manifolds with integral structure (here integral means a natural Spin-structure on  $M$ ). These secondary invariants might be related to the Spin-refinement of the Witten-Reshetikhin-Turaev invariants, defined in [14, 17, 3] as it is the case for the cohomological case (see [9]). Also for  $r = 4$ , the Spin-refinement of the Witten-Reshetikhin-Turaev invariant is equivalent to the Rokhlin invariant. It would be interesting to find a geometric interpretation of the invariants of this paper for  $r = 4$ .

## 1. DECORATED $\mathbb{C}/2\mathbb{Z}$ -SPIN MANIFOLDS

**1.1.  $\mathbb{C}/2\mathbb{Z}$ -spin manifolds.** For a quick overview on classical spin structures, see [15]. Quantum invariants for 3-manifolds with spin structures have been obtained in [3, 14, 17] and extended to TQFTs in [7]. In [4, 5], following a decomposition formula given in [16], the first author considered generalized spin structures with coefficients in  $\mathbb{Z}/d$ ,  $d$  even,

and studied refined quantum invariants of the  $A$  series involving those structures. Similar refinements for general modular categories are developed in [2]. In this paper we will use generalized spin structures whose coefficient group is  $\mathbb{C}/2\mathbb{Z}$  with discrete topology.

**Definition 1.1.** For  $n \geq 2$ , the Lie group  $\text{Spin}(n, \mathbb{C}/2\mathbb{Z})$  is defined by

$$\text{Spin}(n, \mathbb{C}/2\mathbb{Z}) = \frac{(\mathbb{C}/2\mathbb{Z})_{\text{discrete}} \times \text{Spin}(n)}{(\bar{1}, -1)} .$$

This group is a non trivial regular cover of  $\text{SO}(n)$  with Galois group  $\mathbb{C}/2\mathbb{Z}$ . If we replace  $\mathbb{C}/2\mathbb{Z}$  by the discrete circle  $S^1_{\text{discrete}}$  then we would get a group  $\text{Spin}(n, S^1)$  isomorphic to  $\text{Spin}^c(n)$ , but with stronger Lie structure, i.e. the identity  $\text{Spin}(n, S^1) \rightarrow \text{Spin}^c(n)$  is a smooth bijection which is not a diffeomorphism.

Let  $n \geq 2$ , and  $P$  be a  $\text{SO}(n)$  principal bundle over  $B$ . A spin structure with coefficients in  $\mathbb{C}/2\mathbb{Z}$  on  $P$  is a  $\mathbb{C}/2\mathbb{Z}$  regular cover of  $P$  whose restriction to a fiber is equivalent to  $\text{Spin}(n, \mathbb{C}/2\mathbb{Z})$  over  $\text{SO}(n)$ , up to isomorphism. A  $\mathbb{C}/2\mathbb{Z}$  regular cover over  $P$  is classified by a cohomology class  $\sigma \in H^1(P; \mathbb{C}/2\mathbb{Z})$ . For  $n \geq 3$ , the condition on the fiber says that the restriction to the fiber  $i^*(\sigma) \in H^1(\text{SO}(n); \mathbb{C}/2\mathbb{Z}) = \mathbb{Z}/2$  is non zero. For  $n = 2$  the condition on the fiber says that  $i^*(\sigma) = \bar{1} \in H^1(\text{SO}(2); \mathbb{C}/2\mathbb{Z}) = \mathbb{C}/2\mathbb{Z}$ . We get the alternative definition below.

**Definition 1.2** (Cohomological definition). Let  $n \geq 2$ , and  $P$  be a  $\text{SO}(n)$  principal bundle over  $B$ . A  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $P$  is a class  $\sigma \in H^1(P; \mathbb{C}/2\mathbb{Z})$  whose restriction to the fiber has order 2.

The obstruction and parametrization problem for  $\mathbb{C}/2\mathbb{Z}$ -spin structures is encoded in the following exact sequence:

$$0 \rightarrow H^1(B; \mathbb{C}/2\mathbb{Z}) \rightarrow H^1(P; \mathbb{C}/2\mathbb{Z}) \rightarrow H^1(\text{SO}(n); \mathbb{C}/2\mathbb{Z}) \rightarrow H^2(B; \mathbb{C}/2\mathbb{Z}).$$

The obstruction is the second Stiefel-Whitney class  $w_2(P) \in H^2(B; \mathbb{C}/2\mathbb{Z})$ , and if it is non-empty, the set of  $\mathbb{C}/2\mathbb{Z}$ -spin structures supports an affine action of  $H^1(B; \mathbb{C}/2\mathbb{Z})$ .

The definition of  $\mathbb{C}/2\mathbb{Z}$ -spin structures applies to oriented manifolds using the oriented framed bundle. Here the choice of the Riemannian structure is irrelevant.

**Definition 1.3.** A  $\mathbb{C}/2\mathbb{Z}$ -spin manifold is an oriented manifold together with a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on its oriented framed bundle.

We will denote by  $\text{Spin}(M, \mathbb{C}/2\mathbb{Z})$  the set of  $\mathbb{C}/2\mathbb{Z}$ -spin structures on the oriented manifold  $M$ .

**Remark 1.4.** Corresponding to the inclusion map  $\text{SO}(n) = \text{SO}(\mathbb{R}^n) \rightarrow \text{SO}(1+n) = \text{SO}(\mathbb{R} \oplus \mathbb{R}^n)$  we get a stabilization map  $P \rightarrow P \times_{\text{SO}(n)} \text{SO}(n+1)$ . The associated restriction map on the set of  $\mathbb{C}/2\mathbb{Z}$ -spin structures is a bijection. This is used to define a boundary map

$$\partial : \text{Spin}(M, \mathbb{C}/2\mathbb{Z}) \rightarrow \text{Spin}(\partial M, \mathbb{C}/2\mathbb{Z}) .$$

If we have a  $\mathbb{C}/2\mathbb{Z}$ -spin-structure  $\sigma$  which is only defined on a submanifold  $N \subset M$ , then the extending problem gives a relative obstruction  $w_2(M, \sigma) \in H^2(M, N; \mathbb{C}/2\mathbb{Z})$ .

**Definition 1.5.** Let  $M$  be an oriented manifold,  $N \subset M$ , and  $w \in H^2(M, N; \mathbb{C}/2\mathbb{Z})$ . A  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma$  on  $N$  is said to be complementary to  $w$  if and only if  $w_2(M, \sigma) = w$ .

If  $H^0(M, N)$  is trivial, then the set  $\text{Spin}(N, w; \mathbb{C}/2\mathbb{Z})$  of  $\mathbb{C}/2\mathbb{Z}$ -spin structures on  $N$  complementary to  $w$ , if non empty, still supports a natural affine action of  $H^1(M; \mathbb{C}/2\mathbb{Z})$ .

**1.2. Surgery presentation of  $\mathbb{C}/2\mathbb{Z}$ -spin 3-manifold.** Let  $L = (L_1, \dots, L_m)$  be an oriented framed link in  $S^3$  with linking matrix  $B = (B_{ij})_{i,j=1,m}$  and denote by  $S^3(L)$  the 3-manifold obtained by surgery. Then the relative obstruction gives a bijection

$$\psi_L : \text{Spin}(S^3 \setminus L, \mathbb{C}/2\mathbb{Z}) \rightarrow H^2(S^3, S^3 \setminus L; \mathbb{C}/2\mathbb{Z}) \approx (\mathbb{C}/2\mathbb{Z})^m.$$

Recall that a  $\mathbb{C}/2\mathbb{Z}$ -spin structure is a cohomology class on the oriented framed bundle and can be evaluated on a framed circle. A key observation is that evaluation on a trivial simple curve framed by a spanning disc is  $\bar{1} \in \mathbb{C}/2\mathbb{Z}$ . From this we see that  $\sigma \in \text{Spin}(S^3 \setminus L)$  extends to  $S^3(L)$  if and only if  $\psi_L(\sigma) = c = (c_j)_{1 \leq j \leq m}$  satisfies the characteristic equation

$$(1) \quad Bc = (B_{jj})_{1 \leq j \leq m} \pmod{2}.$$

If moreover we have a link  $K = (K_1, \dots, K_\nu)$  in  $S^3 \setminus L$ , and  $\nu$  coefficients  $w_1, \dots, w_\nu$ , representing an element  $w \in H^2(S^3(L), S^3(L) \setminus K; \mathbb{C}/2\mathbb{Z}) \approx H_1(K) \approx (\mathbb{C}/2\mathbb{Z})^\nu$ , then  $\sigma \in \text{Spin}(S^3 \setminus L, \mathbb{C}/2\mathbb{Z})$  extends to an element of  $\text{Spin}(S^3(L) \setminus K, w; \mathbb{C}/2\mathbb{Z})$  if and only if  $\psi_L(\sigma) = c = (c_j)_{1 \leq j \leq m}$  satisfies the equation

$$(2) \quad B(c + c') = (B_{jj})_{1 \leq j \leq m} \pmod{2}$$

where  $c'_j = \sum_{\nu=1}^k w_\nu \text{lk}(L_j, K_\nu)$

Using that restriction maps for  $\mathbb{C}/2\mathbb{Z}$ -spin structures are equivariant with respect to affine actions of cohomologies we get the following proposition:

**Proposition 1.6.** *With the notation above, the map  $\psi_L$  induces a bijection between  $\text{Spin}(S^3(L) \setminus K, w; \mathbb{C}/2\mathbb{Z})$  and the solutions of Equation 2.*

We have obtained a combinatorial description of  $\text{Spin}(S^3(L) \setminus K, w; \mathbb{C}/2\mathbb{Z})$  for  $w \in H^2(S^3(L), S^3(L) \setminus K; \mathbb{C}/2\mathbb{Z})$ .

Theorem 3.9 is a spin version of Kirby theorem that will relate two surgery presentations of the same 3-manifold with complementary  $\mathbb{C}/2\mathbb{Z}$ -spin structure (see also [2]). The appropriate Kirby moves will be obtained in Section 2 by computing the obstructions on the elementary Kirby moves.

**Remark 1.7.** Let  $M$  be a 3-manifold equipped with a  $\text{Spin}(M; \mathbb{C}/2\mathbb{Z})$ -structure  $\sigma$ . Then  $\sigma$  induces a function on isotopy classes of smooth framed oriented simple curve in  $M$  by

$$\sigma(\gamma) = \langle \sigma, [\gamma] \rangle,$$

where  $[\gamma]$  is the integral first homology class of the framed curve  $\gamma$  in the oriented framed bundle of  $M$ . For example,  $\sigma(\text{unknot}) = \bar{1} \in \mathbb{C}/2\mathbb{Z}$ . Then if  $M = S^3 \setminus (L \cup K)$  as above, we have  $c_i = \sigma(m_i) + \bar{1}$  and  $w_j = \sigma(m'_j) + \bar{1}$  where  $m_i$  (resp.  $m'_j$ ) is the standard meridian of the oriented component  $L_i$  (resp.  $K_j$ ). Furthermore,  $\sigma$  extends to  $S^3(L) \setminus K$  if and only if for any  $\gamma$  parallel to a component of  $L$ ,  $\sigma(\gamma) = \bar{1}$ .

**1.3.  $\mathbb{C}$ -colored links.** Let  $q = e^{\frac{i\pi}{r}}$  where  $r \in 4\mathbb{N}^* = \{4, 8, 12, \dots\}$  (a similar version exists for odd  $r$  but the associated topological invariants depend weakly on the spin structure). Recall (see [10]) the  $\mathbb{C}$ -algebra  $\bar{U}_q^H \mathfrak{sl}(2)$  given by generators  $E, F, K, K^{-1}, H$  and relations:

$$\begin{aligned} KEK^{-1} &= q^2 E, & KFK^{-1} &= q^{-2} F, & [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}, & E^r &= 0, \\ HK &= KH, & [H, E] &= 2E, & [H, F] &= -2F, & F^r &= 0. \end{aligned}$$

The algebra  $\overline{U}_q^H \mathfrak{sl}(2)$  is a Hopf algebra where the coproduct, counit and antipode are defined in [10]. Recall that  $V$  is a *weight module* if  $V$  splits as a direct sum of  $H$ -weight spaces and  $q^H = K$  as operators on  $V$ .

The category  $\mathcal{C}$  of weight  $\overline{U}_q^H \mathfrak{sl}(2)$  modules is  $\mathbb{C}/2\mathbb{Z}$  graded (by the weights modulo  $2\mathbb{Z}$ ) that is  $\mathcal{C} = \bigoplus_{\bar{\alpha} \in \mathbb{C}/2\mathbb{Z}} \mathcal{C}_{\bar{\alpha}}$  and  $\otimes : \mathcal{C}_{\bar{\alpha}} \times \mathcal{C}_{\bar{\beta}} \rightarrow \mathcal{C}_{\bar{\alpha}+\bar{\beta}}$ . The simple projective modules of  $\mathcal{C}$  are indexed by the set

$$(3) \quad \ddot{\mathbb{C}} = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}.$$

More precisely, for  $\alpha \in \ddot{\mathbb{C}}$ , the  $r$ -dimensional module  $V_\alpha \in \mathcal{C}_{\overline{\alpha+1}}$  is the irreducible module with highest weight  $\alpha + r - 1$  (be aware of the shift between the index  $\alpha$  of a module and its degree  $\overline{\alpha} + 1$ ). The category  $\mathcal{C}$  is a ribbon category and we let  $F$  be the usual Reshetikhin-Turaev functor from  $\mathcal{C}$ -colored ribbon graphs to  $\mathbb{C}$  (which is given by Penrose graphical calculus). Here the twist acts on  $V_\alpha$  by the scalar

$$\theta_\alpha = q^{\frac{\alpha^2 - (r-1)^2}{2}}.$$

The group of invertible modules of  $\mathcal{C}_0$  is generated by  $\varepsilon$  which is the one dimensional vector space  $\mathbb{C}$  on which  $E, F, K + 1$  and  $H - r$  act by 0. For  $k \in \mathbb{Z}$ , we write  $\varepsilon^k$  for the module  $\varepsilon^{\otimes k}$  on which  $E, F, K - (-1)^k$  and  $H - kr$  act by 0. Remark that in  $\mathcal{C}$ , one has  $V_{\alpha+kr} = V_\alpha \otimes \varepsilon^k$ . Furthermore, if  $V \in \mathcal{C}_{\bar{\alpha}}$  then we have

$$(4) \quad F \left( \begin{array}{c} \varepsilon \quad V \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \varepsilon \end{array} \right) = q^{r\bar{\alpha}} F \left( \begin{array}{c} \varepsilon \quad V \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \varepsilon \end{array} \right), \quad F \left( \begin{array}{c} \varepsilon \quad \varepsilon \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \varepsilon \end{array} \right) = F \left( \begin{array}{c} \varepsilon \quad \varepsilon \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \varepsilon \end{array} \right), \quad F \left( \begin{array}{c} \varepsilon \\ \bigcirc \end{array} \right) = -1.$$

The link invariant underlying our construction is the re-normalized link invariant ([12]) that we recall briefly. The modified dimension is the function defined on  $\{V_\alpha\}_{\alpha \in \ddot{\mathbb{C}}}$  by

$$d(\alpha) = -\frac{r\{\alpha\}}{\{r\alpha\}},$$

where  $\{\alpha\} = 2i \sin \frac{\pi\alpha}{r}$ . Let  $L$  be a  $\mathcal{C}$ -colored oriented framed link in  $S^3$  with at least one component colored by an element of  $\{V_\alpha : \alpha \in \ddot{\mathbb{C}}\}$ . Opening such a component of  $L$  gives a 1-1-tangle  $T$  whose open strand is colored by some  $\alpha \in \ddot{\mathbb{C}}$  (here and after we identify  $\ddot{\mathbb{C}}$  with the set of coloring modules  $\{V_\alpha\}$ ). The Reshetikhin-Turaev functor associates an endomorphism of  $V_\alpha$  to this tangle. As  $V_\alpha$  is simple, this endomorphism is a scalar  $\langle T \rangle \in \mathbb{C}$ . The modified invariant is  $F'(L) = d(\alpha)\langle T \rangle$ .

**Theorem 1.8** ([12], see also [1]). *The assignment  $L \mapsto F'(L) = d(\alpha)\langle T \rangle$  described above is an isotopy invariant of the colored framed oriented link  $L$ .*

**Definition 1.9** (Kirby color). For  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ , let  $\bar{\alpha} \in \mathbb{C}/2\mathbb{Z}$  be its class modulo 2. We say that the formal linear combination of colors

$$\Omega_\alpha = \sum_{k=1}^{r/2} d(\alpha + 2k - 1)[\alpha + 2k - 1]$$

is a Kirby color of degree  $\bar{\alpha}$ .

We can color a link by a formal linear combination of colorings and expand it multilinearly. In [8], an invariant of 3-manifolds equipped with cohomology class where constructed from the data of a relative  $G$ -modular category. The category  $\mathcal{C}$  above failed to be a relative  $G$ -modular category because a constant  $\Delta_+$ , used to compute the invariants,

vanishes. In the spin case, the constant  $\Delta_+$  can be replaced by the following  $\Delta_+^{\text{Spin}}$  whose computation is similar to the one of  $\Delta_+$  in [8, Section 1.2]:

$$\begin{aligned}
 \Delta_+^{\text{Spin}} &= \frac{1}{d(\alpha)} F' \left( \text{Diagram: a circle with a dot inside, labeled } \alpha, \text{ and a loop below it labeled } \Omega_\alpha \right) = \frac{1}{d(\alpha)} \sum_{k=1}^{r/2} d(\alpha + 2k - 1) \frac{-rq^{(\alpha+2k-1)\alpha}}{\theta_\alpha \theta_{\alpha+2k-1}} \\
 &= \frac{r}{\{\alpha\} \theta_0^2} \sum_{k=1}^{r/2} q^{\alpha+\frac{1}{2}} q^{-2(k-1)^2} - q^{-\alpha+\frac{1}{2}} q^{-2k^2} \\
 (5) \quad &= rq^{\frac{1}{2}+(r-1)^2} \frac{1-i}{2} \sqrt{r} = \frac{1-i}{2} (rq)^{\frac{3}{2}}
 \end{aligned}$$

where we used the value of the Gauss sum

$$\sum_{k=-r/2+1}^{r/2} q^{-2k^2} = (1-i)\sqrt{r} = 2 \sum_{k=1}^{r/2} q^{-2k^2} = 2 \sum_{k=1}^{r/2} q^{-2(k-1)^2}.$$

#### 1.4. Compatible triple.

**Definition 1.10** (Compatible triple). Let  $K = K_1 \cup \dots \cup K_\nu$  be a  $\mathcal{C}$ -colored framed oriented link with  $\nu$  components in an oriented 3-manifold  $M$ . Let  $\sigma$  be a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $M \setminus K$  and  $c_\sigma \in (\mathbb{C}/2\mathbb{Z})^\nu$  be the associated  $\mathbb{C}/2\mathbb{Z}$ -coloring of the components of  $K$  ( $\sigma$  is complementary to the cohomology class given by  $c_\sigma$ ). We say that  $(M, K, \sigma)$  is a *compatible triple* if each component  $K_j$  of  $K$  is colored by a module of degree  $c_\sigma(K_j)$ .

**Proposition 1.11.** *Let  $(S^3, L \cup K, \sigma)$  be a compatible triple where at least one component of  $L$  is colored by an element of  $\check{\mathbb{C}}$  and the component  $K$  is an oriented ribbon knot colored by  $\varepsilon$ . Then  $\sigma$  extends uniquely to a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $S^3 \setminus L$  and  $(S^3, L, \sigma)$  is a compatible triple. Furthermore,*

$$F'(L \cup K) = q^{r\sigma(K)} F'(L).$$

*Proof.* The obstruction to extending  $\sigma$  to  $S^3 \setminus L$  is given by  $c_\sigma(K)$ . But the compatibility condition for  $K$  implies that  $c_\sigma(K) = \bar{0}$  because  $\varepsilon \in \mathcal{C}_{\bar{0}}$ . Then one can compute  $\sigma(K)$  using some skein relations in  $S^3 \setminus L$ . But these skein relations are precisely given by

$$\begin{aligned}
 \sigma(K) \left( \text{Diagram: crossing of two strands, top-left labeled } K, \text{ top-right labeled } L_i \right) &= \sigma(K) \left( \text{Diagram: crossing of two strands, top-left labeled } K, \text{ top-right labeled } L_i \right) + c_\sigma(L_i), \\
 \sigma(K) \left( \text{Diagram: crossing of two strands, top-left labeled } K, \text{ top-right labeled } K' \right) &= \sigma(K) \left( \text{Diagram: two parallel strands, top-left labeled } K, \text{ top-right labeled } K' \right), \quad \sigma(K) \left( \text{Diagram: a circle labeled } K \right) = \bar{1}.
 \end{aligned}$$

Comparing with the skein relations (4), we get the announced equality. 1.11

**Corollary 1.12.** *Let  $L \cup K_\alpha$  be a link in  $S^3$  with a component colored by  $\alpha \in \check{\mathbb{C}}$  and let  $L \cup K_{\alpha+nr}$  be the same link except that the color  $\alpha$  is changed to  $\alpha + nr$ . Then*

$$F'(L \cup K_{\alpha+nr}) = q^{nr\sigma(K_\parallel)} F'(L \cup K_\alpha).$$

where  $K_\parallel$  is a framed curve parallel to  $K_\alpha$ .

*Proof.* The proof is by induction on  $n$  using  $V_{\alpha+r} = V_\alpha \otimes \varepsilon$  and the fact that the RT functor  $F$  does not distinguish between a strand colored by  $V \otimes V'$  and two parallel strands colored by  $V$  and  $V'$ . Then one can use Proposition 1.11. 1.12

### 1.5. Link presentation of compatible triple.

**Definition 1.13.** A link presentation is a triple  $(L, K, \sigma)$  where

- (1)  $L$  is an oriented framed link in  $S^3$ ,
- (2)  $K$  is a  $\mathcal{C}$ -colored oriented framed link in  $S^3 \setminus L$ ,
- (3)  $\sigma$  is a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $S^3 \setminus (L \cup K)$ ,
- (4) the triple  $(S^3 \setminus L, K, \sigma)$  is compatible,
- (5)  $\sigma$  extends to a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $S^3(L) \setminus K$ .

Link presentations are regarded up to ambient isotopy. We now give several important remarks on this definition:

**Remark 1.14.**

- (1) By the process of surgery on  $L$ , a link presentation gives rise to a compatible triple  $(S^3(L), K, \sigma)$ . Reciprocally, if  $\sigma$  is the restriction of any compatible  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $S^3(L) \setminus K$  then  $\sigma$  clearly satisfies (4) and (5).
- (2) The  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma$  on  $S^3 \setminus (L \cup K)$  induces a  $\mathbb{C}/2\mathbb{Z}$ -coloring  $c_\sigma$  of the components of  $L \cup K$  which by (4) is compatible with the  $\mathcal{C}$ -coloring of  $K$ . Furthermore, Proposition 1.6 implies that  $c_\sigma$  determines  $\sigma$ .
- (3) According to Remark 1.7, the last item is equivalent to the fact that if  $\gamma$  is a parallel to any component of  $L$ , then  $\sigma(\gamma) = \bar{1}$ . It is also equivalent by Proposition 1.6, to the fact that  $c$  satisfies Equation 2 where  $c_i = c_\sigma(L_i)$  and  $w_j = c_\sigma(K_j)$ .

**Definition 1.15.** A link presentation  $(L, K, \sigma)$  is *computable* if

- either  $L = \emptyset$  and  $K$  has a component colored by some  $\alpha \in \ddot{\mathbb{C}}$ ,
- or for every component  $L_i$  of  $L$ ,  $c_\sigma(L_i) \notin \mathbb{Z}/2\mathbb{Z}$ .

**Proposition 1.16.** *Let  $(L, K, \sigma)$  be a computable link presentation. We color every component of  $L$  by a Kirby color of degree  $c_\sigma(L_i)$  and compute its image by  $F'$ . Then the complex number  $F'(L, K, \sigma)$  obtained by this process is independent of the choice of the Kirby colors.*

*Proof.* If a component  $L_i$  of  $L$  is colored by  $d(\alpha)V_\alpha$  then changing it to  $d(\alpha + nr)V_{\alpha+nr}$  does not affect  $F'$ . Indeed, by Remark 1.14(3), and by Corollary 1.12 the invariant changes by the sign  $q^{rn} = (-1)^n$ . But  $d(\alpha + nr) = (-1)^n d(\alpha)$  so the two possible signs compensate. Hence changing the Kirby color used to color  $L_i$  from  $\Omega_\alpha$  to

$$\Omega_{\alpha+2} = \Omega_\alpha - d(\alpha + 1)[\alpha + 1] + d(\alpha + r + 1)[\alpha + r + 1]$$

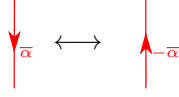
does not change the invariant.

Remark for later that if, instead of  $\Omega_\alpha$ , one colors a component with the formal combination  $\tilde{\Omega}_\alpha = \frac{1}{2} \sum_{k=1}^r d(\alpha + 2k - r - 1)[\alpha + 2k - r - 1]$ , it does not change the invariant. 1.16

## 2. MOVES ON LINK PRESENTATIONS

Here we present five moves on link presentations, then in Section 3 we will use these moves to show that the invariant is well defined. In the following pictures involving link presentations, the surgery link  $L$  is colored in red and the  $\mathcal{C}$ -colored link  $K$  is colored in blue. A black strand can be any component of  $K \cup L$ . The first three moves are the spin version of the usual Kirby moves. As remarked by Gompf and Stipsicz (see [13, Chapter 5]), every Kirby move describes a canonical isotopy class of diffeomorphisms between the surgered 3-manifolds. The purpose of the fourth and fifth moves is to create a generically colored knot in the 3-manifold.

### 2.1. Orientation change.



**Definition 2.1** (Orientation move). The link presentations  $(L, K, \sigma)$  and  $(L', K, \sigma)$  are related by an orientation move if  $L = L'$  as an unoriented framed link but the orientation of the components of  $L'$  might differ from those of  $L$ . Remark that the  $\mathbb{C}/2\mathbb{Z}$ -spin structure does not change but  $c_\sigma(L_i) = -c_\sigma(L'_i)$  if the orientation of  $L_i$  has changed.

**Proposition 2.2.** *If two link presentations  $(L, K, \sigma)$  and  $(L', K, \sigma)$  are related by an orientation move and if they are computable then*

$$F'(L, K, \sigma) = F'(L', K, \sigma)$$

*Proof.* The following changes does not affect the invariant  $F'$  of  $\mathcal{C}$ -colored link:

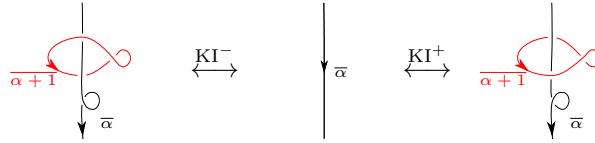
- change a  $\mathcal{C}$ -color  $V$  of  $L$  with an isomorphic module,
- change the orientation of a component of  $L$  and simultaneously change its  $\mathcal{C}$ -color with its dual.

Hence the proposition follows from the fact that for  $\alpha \in \check{\mathbb{C}}$ ,  $V_\alpha^* \simeq V_{-\alpha}$  and  $\Omega_\alpha^* \simeq \Omega_{-\alpha}$ .

2.2

**Remark 2.3.** The manifold obtained by surgery does not depends of the orientation of the surgery link.

**2.2. Stabilization.** The usual Kirby I move is a stabilization that consists in adding to  $L$  an unknot with framing  $\pm 1$ . This move always leads to a non computable presentation. For this reason, we are led as in [8] to introduce a modified stabilization.



**Definition 2.4** ( $KI^\pm$ -move). The link presentation  $(L' \cup o, K', \sigma')$  is obtained from the link presentation  $(L, K, \sigma)$  by a positive  $KI^\pm$ -move if  $o$  is an unknot with framing  $\pm 1$  obtained by adding a full positive or negative twist to a meridian of a component  $J$  of  $L \cup K$ . We orient this meridian so that its linking number with  $J$  is  $\mp 1$ . The only changed component of  $L \cup K$  is  $J$  whose corresponding component  $J'$  of  $L' \cup K'$  is obtained by adding a full twist to  $J$ . The  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma'$  is determined by  $c_{\sigma'}|_{L \cup K} = c_\sigma$  and  $c_{\sigma'}(o) = c_\sigma(J) + \bar{1}$ .

We say that  $(L, K, \sigma)$  is obtained from  $(L \cup o, K, \sigma')$  by a negative  $KI^\pm$ -move.

**Proposition 2.5.** *A  $KI^\pm$ -move between  $(L, K, \sigma)$  and  $(L' \cup o, K', \sigma')$  induces a canonical (up to isotopy) diffeomorphism  $(S^3(L), K, \sigma) \simeq (S^3(L' \cup o), K', \sigma')$ .*

*Proof.* The  $KI$ -move does not change  $L \cup K$  outside a tubular neighborhood  $T$  of  $J$ . There is a canonical diffeomorphism  $f : (S^3(L), K) \xrightarrow{\sim} (S^3(L'), K')$ . Hence there exists a  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma'$ , which is the image of  $\sigma$  by  $f$ , such that  $(S^3(L'), K', \sigma')$  is a compatible triple. Furthermore, the diffeomorphism  $f$  is the identity outside  $T$ . Thus  $(L', K', \sigma')$  is a link presentation, furthermore  $f$  send meridian of the components of  $L \cup K$  to the corresponding meridian of the components of  $L' \cup K'$  thus  $c_{\sigma'}|_{L \cup K} = c_\sigma$ . Finally, the new component has framing  $\pm 1$  and is only linked  $\mp 1$  times with  $J'$  so Equation (2) implies that  $c_{\sigma'}(o) = c_\sigma(J) + \bar{1}$ .

2.5

**Proposition 2.6.** *If  $(L \cup o, K, \sigma')$  is obtained from  $(L, K, \sigma)$  by a positive  $KI^\pm$ -move and if they are both computable, then*

$$F'(L \cup o, K, \sigma') = \Delta_{\pm}^{\text{Spin}} F'(L, K, \sigma).$$

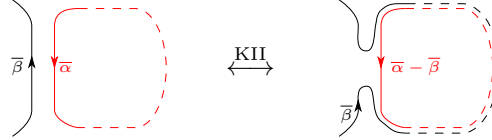
where  $\Delta_+^{\text{Spin}} = (1 - i)r^{\frac{3}{2}}q^{\frac{3}{2}}$  and  $\Delta_-^{\text{Spin}} = (1 + i)r^{\frac{3}{2}}q^{-\frac{3}{2}}$  is the conjugate complex number.

*Proof.* We give the proof for a  $KI^+$ -move. The other case is similar. Let  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ , since

$V_\alpha$  is simple  $F \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)$  is a scalar endomorphism of  $V_\alpha$ . From Equation (5) this scalar is

equal to  $\Delta_+^{\text{Spin}}$ . Moreover, this scalar times  $d(\alpha)$  is equal  $F'$  of the braid closure of this 1-1 tangle. Now as before,  $\Delta_+^{\text{Spin}}$  does not change if we replace  $\Omega_\alpha$  by any Kirby color of degree  $\bar{\alpha}$ . Furthermore,  $\Delta_+^{\text{Spin}}$  is independent of  $\alpha$  so this scalar is the same for any simple module of  $\mathcal{C}_{\alpha+1}$ . As  $\mathcal{C}_{\alpha+1}$  is semi-simple, if we replace  $V_\alpha$  with any module  $V \in \mathcal{C}_{\alpha+1}$ , then this tangle evaluates to  $\Delta_+^{\text{Spin}} \text{Id}_V$ . It follows that whatever the value of  $c_\sigma(J) \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$  is we have  $F'(L \cup o, K, \sigma') = \Delta_+^{\text{Spin}} F'(L, K, \sigma)$ . 2.6

### 2.3. Handle slide.



**Definition 2.7** (KII-move). The link presentations  $(L, K, \sigma)$  and  $(L', K', \sigma')$  are related by a KII-move if  $L_i$  is a component of  $L$  which is different from a component  $J$  of  $L \cup K$  and  $J'$  is a component of  $L' \cup K'$  such that  $L' \setminus J' = L \setminus J$  and  $J'$  is a connected sum of  $J$  with a parallel copy of  $L_i$ . If  $J$  is a component of  $K$ , then  $J'$  and  $J$  have the same  $\mathcal{C}$ -color. The  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma'$  is determined by  $c_{\sigma'| (L \cup K') \setminus L'_i} = c_{\sigma| (L \cup K) \setminus L_i}$  and  $c_{\sigma'}(L'_i) = c_\sigma(L_i) - c_\sigma(J)$ .

**Proposition 2.8.** *A KII-move between  $(L, K, \sigma)$  and  $(L', K', \sigma')$  induces a canonical (up to isotopy) diffeomorphism  $(S^3(L), K, \sigma) \simeq (S^3(L'), K', \sigma')$ .*

*Proof.* The KII-move does not change  $L \cup K$  outside a genus 2 handlebody  $H$  embedded in  $S^3$  formed by a tubular neighborhood of  $J$ , of  $L_i$  and of the path from  $J$  to the parallel copy of  $L_i$  used to make the connected sum. There is a canonical diffeomorphism  $f : (S^3(L), K) \xrightarrow{\sim} (S^3(L'), K')$ . Hence there exists a  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma'$ , which is the image of  $\sigma$  by  $f$ , such that  $(S^3(L'), K', \sigma')$  is a compatible triple. The diffeomorphism  $f$  is the identity outside  $H$ , so  $(L', K', \sigma')$  is a presentation. Furthermore,  $f$  sends meridians to the components of  $L \cup K$  to the corresponding meridians of the components of  $L' \cup K'$  except for the meridian of  $L_i$  which is sent to the connected sum of the meridian of  $L'_i$  with the meridian of  $J'$ . Hence  $\sigma(m_{L_i}) = \sigma'(m_{L'_i}) + \sigma'(m_{J'}) + \bar{1}$  where the  $\bar{1}$  comes from the connected sum. This leads to the announced formula for  $c_{\sigma'}$ . 2.8

**Proposition 2.9.** *If two link presentations  $(L, K, \sigma)$  and  $(L', K', \sigma')$  are related by a KII-move and if they are both computable then*

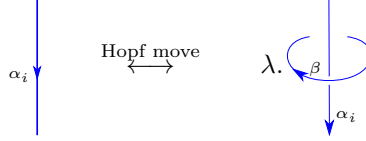
$$F'(L, K, \sigma) = F'(L', K', \sigma')$$

*Proof.* First the remark in the proof of Proposition 1.16 implies that instead of using a Kirby color of degree  $\Omega_\alpha$  we can compute the image by  $F'$  of the link presentation using



the Kirby color  $\frac{1}{2}\tilde{\Omega}_\alpha = \frac{1}{2} \sum_{k=1}^r d(\alpha + 2k - r - 1)[\alpha + 2k - r - 1]$ . Then the proof is the same that in [8, Lemma 5.9] where the module  $\varepsilon$  used in [8] correspond to  $\varepsilon^2$  in this paper. 2.9

#### 2.4. Hopf stabilization.



**Definition 2.10** (Hopf move). The link presentation  $(L, K \cup o, \sigma')$  is obtained from the link presentation  $(L, K, \sigma)$  by a positive Hopf move if  $o$  is a zero framed meridian of a component  $K_i$  of  $K$ . The color  $\alpha_i$  of  $K_i$  has to be in  $\check{\mathbb{C}}$  and the color of the newly added component is the coefficient  $\lambda = \frac{d(\alpha_i)}{-rq^{\beta\alpha_i}}$  times a color  $\beta \in \check{\mathbb{C}}$ .

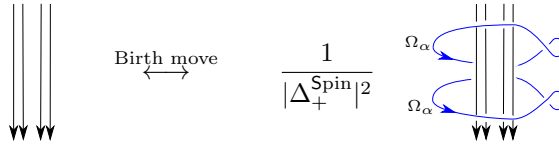
**Proposition 2.11.** *If two link presentations  $(L, K, \sigma)$  and  $(L, K \cup o, \sigma')$  are related by a Hopf stabilization then*

$$F'(L, K, \sigma) = F'(L, K \cup o, \sigma').$$

*Proof.* The simplicity of  $V_\alpha$  for  $\alpha \in \check{\mathbb{C}}$  implies that  $F\left(\begin{array}{c} \text{blue loop} \\ \beta \end{array}\right)$  is a scalar endomorphism of  $V_\alpha$ . This scalar is computed (for example in [12]) and is given by  $\frac{-rq^{\beta\alpha_i}}{d(\alpha_i)}$ . It follows that  $F'(L, K \cup o, \sigma')$  is just  $\frac{-rq^{\beta\alpha_i}}{d(\alpha_i)} \lambda$  times  $F'(L, K, \sigma)$ . 2.11

**Remark 2.12.** There is a similar notion of Hopf stabilization for compatible triple  $(M, K, \sigma)$ . The resulting compatible triple  $(M, K \cup o, \sigma')$  can be seen as a banded connected sum of  $(M, K, \sigma)$  with  $(S^3, H, \sigma_H)$  where  $H$  is proportional to a Hopf link in  $S^3$ . Furthermore, a Hopf stabilization of a surgery presentation of a compatible triple is clearly a surgery presentation of a Hopf stabilization of the triple.

**2.5. Birth move.** In the following positive move two new components appear in the  $K$  part of a link presentation.



**Definition 2.13** (Birth move). The link presentation  $(L, K \cup K_+ \cup K_-, \sigma')$  is obtained from the computable link presentation  $(L, K, \sigma)$  by a *positive birth move* if it is given by the following process: Let  $D$  be an oriented disc in  $S^3$  and  $\partial D$  its oriented framed boundary. We assume that  $\partial D$  is in general position relatively to  $L \cup K$  and that  $\bar{\alpha} = \sigma(\partial D) \notin \mathbb{Z}/2\mathbb{Z}$ . Then  $K_+$  and  $K_-$  are two parallel copies of  $\partial D$  with framing  $+1$  and  $-1$  respectively, with the orientation of  $K_+$  reversed and they are both colored by  $\frac{1}{|\Delta_+^{\text{Spin}}|}$  times a Kirby color of degree  $\bar{\alpha}$ . The  $\mathbb{C}/2\mathbb{Z}$ -spin structure  $\sigma'$  is determined by  $(c_{\sigma'})|_{L \cup K} = c_\sigma$  and  $c_{\sigma'}(K_\pm) = \bar{\alpha}$ .

**Proposition 2.14.** *If two computable link presentations  $(L, K, \sigma)$  and  $(L, K \cup K_+ \cup K_-, \sigma')$  are related by a birth move then*

$$F'(L, K, \sigma) = F'(L, K \cup K_+ \cup K_-, \sigma')$$

*Proof.* We need the use of ribbon graphs with coupons. We start with the computable link presentations  $(L, K, \sigma)$  and color the components of  $L_i$  with Kirby colors. Up to isotopy, we can assume that the disc  $D$  is in an horizontal plane and intersects  $n$  vertical strands colored by some modules  $V_1, \dots, V_n$ . These  $n$  strands represent the identity of  $W = V'_1 \otimes \dots \otimes V'_n$  where  $V'_i$  is  $V_i$  or  $V_i^*$  according to the orientation of the  $i^{th}$  strand. By hypothesis,  $W \in \mathcal{C}_{\bar{\alpha}}$ . Then the following modifications does not change the value of the invariant by  $F'$ :

- (1) Replace a neighborhood of  $D$  by a unique vertical strand colored by  $W$  and connected to the  $n$ -strands above and ahead the disc  $D$  by two coupons colored by the morphism  $\text{Id}_W$ .
- (2) Make a  $\text{KI}^+$ -move followed by a  $\text{KI}^-$ -move on the  $W$ -colored strand. Here we directly color the new components  $K_{\pm}$  with a Kirby color divided by  $|\Delta_+^{\text{Spin}}|$ . The two changes of the framing of the  $W$ -colored strand compensate.
- (3) Remove the  $W$ -colored strand and the two coupons and replace it back with the  $n$  original strands.

This sequence of modifications lead to  $F'(L, K \cup K_+ \cup K_-, \sigma')$  as describe by a birth move. 2.14

### 3. THE INVARIANT

**3.1. Main theorem.** In this subsection we define the invariant and give some of its properties. The proofs are postponed until Subsection 3.2.

Recall that if  $F_K M$  is the framed oriented bundle of  $M \setminus K$ , a  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $M \setminus K$  is a cohomology class  $\sigma$  in  $H^1(F_K M; \mathbb{C}/2\mathbb{Z})$ . We will say that  $\sigma$  is *integral* if  $\sigma$  belongs to  $\text{Hom}(H_1(F_K M; \mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$ , via the universal coefficients theorem:  $H^1(F_K M; \mathbb{C}/2\mathbb{Z}) \cong \text{Hom}(H_1(F_K M; \mathbb{Z}); \mathbb{C}/2\mathbb{Z})$ . This means that  $\sigma$  is a natural spin structure on  $M \setminus K$  (associated to the group  $\text{Spin}(n) = \text{Spin}(n, \mathbb{Z}/2\mathbb{Z})$ ).

**Definition 3.1.** A compatible triple  $(M, K, \sigma)$  is admissible if  $K$  has a component colored by  $\alpha \in \check{\mathbb{C}}$  or if  $\sigma$  is a non-integral  $\mathbb{C}/2\mathbb{Z}$ -spin structure on  $M \setminus K$ .

**Theorem 3.2.** *Let  $(M, K, \sigma)$  be an admissible triple.*

- (1) *If  $\sigma$  is not integral then there exists a computable surgery presentation  $(L, K, \sigma)$  of  $(M, K, \sigma)$ . If  $\sigma$  is integral, there exists a computable surgery presentation  $(L, K', \sigma')$  of an Hopf stabilization of  $(M, K, \sigma)$ .*
- (2) *In both cases of part (1) we have,*

$$\mathbf{N}(M, K, \sigma) = \Delta(L)F'(L, K, \sigma) \quad (\text{respectively } \mathbf{N}(M, K, \sigma) = \Delta(L)F'(L, K', \sigma'))$$

*is an invariant of the diffeomorphism class of  $(M, K, \sigma)$ , where  $\Delta(L) = (\Delta_+^{\text{Spin}})^{-b_+} (\Delta_-^{\text{Spin}})^{-b_-}$  with  $(b_+, b_-)$  being the signature of the linking matrix of  $L$ .*

The proof of this theorem will be given in the next section. We now state some properties of the invariant  $\mathbf{N}$ .

If two admissible triple  $(M, K \cup K_{\alpha}, \sigma)$  and  $(M', K' \cup K'_{\alpha}, \sigma')$  have a distinguished component colored by the same  $\alpha \in \check{\mathbb{C}}$ , then one can consider their banded connected sum  $(M, K \cup K_{\alpha}, \sigma) \#_{\alpha} (M', K' \cup K'_{\alpha}, \sigma') = (M \# M', K \cup (K_{\alpha} \# K'_{\alpha}), \sigma \cup \sigma')$  obtained by removing

from  $M$  a small 3-ball intersecting  $K_\alpha$ , removing from  $M'$  a small 3-ball intersecting  $K'_\alpha$  and gluing these two manifolds along their diffeomorphic sphere. Then we have

**Proposition 3.3** (Banded connected sum). *Under the above hypothesis we have*

$$\mathbf{N}((M, K \cup K_\alpha, \sigma) \#_\alpha (M', K' \cup K'_\alpha, \sigma')) = d(\alpha)^{-1} \mathbf{N}(M, K \cup K_\alpha, \sigma) \mathbf{N}(M', K' \cup K'_\alpha, \sigma').$$

For the ordinary connected sum of compatible triples, remark that the result of the connected sum of an admissible triple  $(M, K, \sigma)$  with any compatible  $(M', K', \sigma')$  is admissible. This allows to compute a secondary invariant  $\mathbf{N}^0$  for non admissible triples:

**Theorem 3.4** ( $\mathbf{N}^0$  and connected sum). *There exists a unique  $\mathbb{C}$ -valued invariant  $\mathbf{N}^0$  defined on any compatible triple (not necessary admissible) which is zero on admissible triples and such that for all admissible triple  $(M, K, \sigma)$  and any not necessary admissible triple  $(M', K', \sigma')$ ,*

$$\mathbf{N}((M, K, \sigma) \# (M', K', \sigma')) = \mathbf{N}(M, K, \sigma) \mathbf{N}^0(M', K', \sigma').$$

**3.2. Proof of the invariance.** The moves of Section 2 can be considered as the edges a graph  $\Gamma$  whose vertices are link presentations. Let  $\Gamma_0$  be the subgraph of  $\Gamma$  consisting of computable presentations. Then the propositions of Section 2 imply that

**Proposition 3.5.** *The function  $(L, K, \sigma) \mapsto \Delta(L)F'(L, K, \sigma)$  is constant on the connected components of  $\Gamma_0$ .*

*Proof.* The factors  $\Delta(L)$  and  $F'(L, K, \sigma)$  are invariant by all the moves except for  $\text{KI}^\pm$  moves for which they exactly compensates. Indeed, a positive  $\text{KI}^\pm$  move adds exactly 1 to the signature integer  $b_\pm$ . 3.5

*Proof Theorem 3.2 Part (1).* Here we prove the existence of computable presentations. Let  $(M, K, \sigma)$  be admissible as in Theorem 3.2 Part (1). If  $\sigma$  is integral, then  $K$  has a component colored by an element of  $r\mathbb{Z} \subset \check{\mathbb{C}}$ . Then we replace  $(M, K, \sigma)$  by an Hopf stabilization on such an edge so that the added meridian has a color in  $\check{\mathbb{C}} \setminus \mathbb{Z}$ .

Now we are reduced to the case where  $\sigma$  has a non integral value. Consider any surgery presentation  $(L, K, \sigma)$  of  $(M, K, \sigma)$  and consider a component  $J$  of  $L \cup K$  with  $c_\sigma(J) \notin \mathbb{Z}/2\mathbb{Z}$ . If  $c_\sigma(L_i) \in \mathbb{Z}/2\mathbb{Z}$  for some component  $L_i$  of  $L$  then we do a KII move by sliding  $J$  on  $L_i$ . In the resulting link  $L'$ , we have  $c_\sigma(L'_i) \notin \mathbb{Z}/2\mathbb{Z}$  where  $L'_i$  corresponds to  $L_i$  and the other values of  $c_\sigma$  are unchanged. Repeating this for all components of  $L$  if necessary leads to a computable presentation of  $(M, K, \sigma)$ . 3.2(1)

**Remark 3.6.** It is clear that if  $(M_1, K_1, \sigma_1)$  and  $(M_2, K_2, \sigma_2)$  are two Hopf stabilizations of  $(M, K, \sigma)$ , then there exists  $(M_{12}, K_{12}, \sigma_{12})$  which is both a Hopf stabilization of  $(M_1, K_1, \sigma_1)$  and  $(M_2, K_2, \sigma_2)$ .

With this remark, Theorem 3.2 Part (2) follows from the following proposition whose proof is essentially a reformulation of [8]:

**Proposition 3.7.** *Any two computable link presentations of an admissible triple  $(M, K, \sigma)$  correspond to vertices of the same connected component of  $\Gamma_0$ .*

*Proof.* The key point is Kirby's theorem and its refinement:

**Theorem 3.8** (see Theorem 5.2 of [8]). *Let  $L \cup K$  and  $L' \cup K'$  be two link in  $S^3$  and  $f : (S^3(L), K) \rightarrow (S^3(L'), K')$  be an orientation preserving diffeomorphism. Then there is a sequence of orientation moves,  $\text{KI}^\pm$  moves and KII moves transforming  $(L, K)$  to  $(L', K')$  whose associated canonical diffeomorphism is  $f$ .*

We can use this theorem in the case a diffeomorphism  $g : (M, K, \sigma) \rightarrow (M', K', \sigma')$  respects the  $\mathbb{C}/2\mathbb{Z}$ -spin structure. Then if  $(L, K, \sigma)$  is a presentation of  $(M, K, \sigma)$  and  $(L', K', \sigma')$  is a presentation of  $(M', K', \sigma')$ ,  $g$  induces a diffeomorphism  $f$  as in Theorem 3.8 that will send the  $\mathbb{C}/2\mathbb{Z}$ -spin structure of  $S^3(L) \setminus K$  to the  $\mathbb{C}/2\mathbb{Z}$ -spin structure of  $S^3(L') \setminus K'$ . As a consequence, there exists a sequence of orientation moves,  $KI^\pm$  moves and  $KII$  moves connecting the link presentations  $(L, K, \sigma)$  and  $(L', K', \sigma')$  (with their  $\mathbb{C}/2\mathbb{Z}$ -spin structure). As a corollary, we have

**Theorem 3.9.**  *$(L, K, \sigma)$  and  $(L', K', \sigma')$  are link presentations of positively diffeomorphic compatible triples if and only if they are connected by a finite sequence of orientation moves,  $KI^\pm$  moves and  $KII$  moves.*

Let  $\bar{L} = (L, K, \sigma)$  and  $\bar{L}' = (L', K', \sigma')$  be two computable link presentations of the same admissible triple. By the Theorem 3.9, there is a finite sequence of orientation moves,  $KI^\pm$  moves and  $KII$  moves connecting  $\bar{L}$  and  $\bar{L}'$ .

To prove Theorem 3.2 Part (2), we first reduce to the case when  $K$  has a component colored by an element of  $\ddot{\mathbb{C}}$ : If it is not the case, then there is a component  $L_i$  of  $L$  with  $c_\sigma(L_i) \notin \mathbb{Z}/2\mathbb{Z}$ . We consider a small disc in  $S^3$  that intersects  $L_i$  once. We do a birth move on this disc that creates two new components. Then we can perform the analog of the finite sequence of orientation moves,  $KI^\pm$  moves and  $KII$  moves on this link presentation and this leads to a link presentation related to  $\bar{L}'$  by a death move (the two created components stay parallel and unknotted in  $S^3$  during all this process). Hence up to replacing  $\bar{L}$  and  $\bar{L}'$  by these computable presentations obtained from them by a birth move, we can assume that  $K$  has a component (say  $K_1$ ) colored by a element of  $\ddot{\mathbb{C}}$ .

Second, we do two Hopf stabilizations on  $K_1$ , creating two meridians  $J_\alpha$  and  $J_\beta$  colored with some generic color  $\alpha, \beta \in \ddot{\mathbb{C}}$  (by generic, we mean that  $\alpha$  and  $\beta$  are rationally independent with all the values of  $\sigma$  or  $c_\sigma$ ). Now sliding  $J_\alpha$  and  $J_\beta$  on the surgery components allows us to change arbitrarily the  $\mathbb{C}/2\mathbb{Z}$ -coloring of surgery components by adding any element of  $\mathbb{Z}\alpha + \mathbb{Z}\beta$  through computable presentation (this is the reason of the use of two meridians with independent colors).

Now we start performing the analog of the finite sequence of orientation moves,  $KI^\pm$  moves and  $KII$  moves ignoring  $J_\alpha$  and  $J_\beta$  but during the sequence,

- if a  $KII$  move which is a sliding on  $L_i$  generates an integral value of  $c_\sigma(L'_i)$ , we first change the color of  $L_i$  by an element of  $\mathbb{Z}\alpha + \mathbb{Z}\beta$  so that the  $KII$  move does not cause the appearance of a color in  $\mathbb{Z}/2\mathbb{Z}$  ;
- if a  $KI^\pm$  move is performed around an edge of  $K$  of integral degree, we do it instead on  $J_\alpha$  and then slide the edge of  $K$  on the created component.

Doing this we follow a path in  $\Gamma_0$  leading to  $\bar{L}'$  union two knots  $J_\alpha$  and  $J_\beta$  that may be linked with other components of  $L' \cup K'$ . But in  $S^3(L') \setminus K'$ ,  $J_\alpha$  and  $J_\beta$  are meridians of a component (say  $K'_1$ ) of  $K'$ . Thus there exists an isotopy in  $S^3(L') \setminus K'$  moving  $J_\alpha$  and  $J_\beta$  to a small neighborhood of  $K'_1$ . To this isotopy corresponds a sequence of  $KII$  moves where only  $J_\alpha$  and  $J_\beta$  are sliding on the surgery components. This sequence leads to a double Hopf stabilization of  $\bar{L}'$ . Furthermore, as a color of a surgery component  $L'_i$  during these moves belongs to  $c_{\sigma'} + \alpha\mathbb{Z} + \beta\mathbb{Z}$  modulo 2 it is never an integer and all the sequence is in  $\Gamma_0$ . 3.7

## REFERENCES

1. Y. Akutsu, T. Deguchi, and T. Ohtsuki - *Invariants of colored links*. J. Knot Theory Ramifications **1** (1992), no. 2, 161–184.

2. A. Beliakova, C. Blanchet, E. Contreras - In progress.
3. C. Blanchet - *Invariants of 3-manifolds with spin structure*. Comm. Math. Helv. 67 (1992), 406-427.
4. C. Blanchet - *Hecke algebras, modular categories and 3-manifolds quantum invariants* - Topology 39 (2000) 193-223.
5. C. Blanchet - *A spin decomposition of the Verlinde formulas for type A modular categories* - Comm. in Math. Physics, Vol. 257, N. 1, (2005), 1-28.
6. C. Blanchet, F. Costantino, N. Geer, B. Patureau-Mirand - *Non semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants*, arXiv:1404.7289.
7. C. Blanchet, G. Masbaum - *Topological quantum field theories for surfaces with spin structure*, Duke Mathematical Journal, 82 (1996), 229-267.
8. F. Costantino, N. Geer, B. Patureau-Mirand - *Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories*, To appear in Journal of Topology, arXiv:1202.3553.
9. F. Costantino, N. Geer, B. Patureau-Mirand - *Relations between Witten-reshetikhin-turaev and non semi-simple  $\mathfrak{sl}(2)$  3-manifold invariants*, arXiv:1310.2735.
10. F. Costantino, N. Geer, B. Patureau-Mirand - *Some remarks on a quantization of  $\mathfrak{sl}(2)$* , in preparation.
11. F. Costantino, J. Murakami - *On  $SL(2, \mathbb{C})$  quantum 6j-symbols and their relation to the hyperbolic volume*, Quantum Topology 4, (2013), no. 3, 303-351.
12. N. Geer, B. Patureau-Mirand, V. Turaev - *Modified quantum dimensions and re-normalized link invariants*. Compos. Math. 145 (2009), no. 1, 196-212.
13. R. Gompf, A. Stipsicz - *4-manifolds and Kirby calculus*. Amer. Math. Soc. Providence (RI), 1999, xv+558 pages.
14. R. Kirby, P. Melvin - *The 3-manifold invariants of Witten and Reshetikhin-Turaev for  $\mathfrak{sl}(2, \mathbb{C})$* . Invent. Math. 105, (1991), 473-545.
15. J. Milnor - *Spin structures on manifolds* Enseign. Math., II. 9 (1963), 198-203.
16. H. Murakami - *Quantum invariants for 3-manifolds*, in : Proc. Appl. Math. Workshops, 4, The 3rd Korea-Japan School of Knots and Links, (1994) (eds. K.H. Ko and G.T. Jin), 129-143.
17. V.G. Turaev. *State sum models in low-dimensional topology*, Proc. ICM Kyoto **121** Vol. 1 (1990), 689-698.
18. V.G. Turaev - *Quantum invariants of knots and 3-manifolds*. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, (1994).
19. E. Witten - *Quantum field theory and Jones polynomial*. Comm. Math. Phys. 121 (1989), 351-399.

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